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2005 J. Phys. A: Math. Gen. 38 L79

(http://iopscience.iop.org/0305-4470/38/6/L02)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) L79–L86 doi:10.1088/0305-4470/38/6/L02

LETTER TO THE EDITOR

Entanglement and tensor product decomposition for two fermions

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Received 26 November 2004
Published 26 January 2005
Online at stacks.iop.org/JPhysA/38/L79

Abstract
The problem of the choice of tensor product decomposition in a system of two fermions with the help of Bogoliubov transformations of creation and annihilation operators is discussed. The set of physical states of the composite system is restricted by the superselection rule forbidding the superposition of fermions and bosons. It is shown that the Wootters concurrence is not the proper entanglement measure in this case. The explicit formula for the entanglement of formation is found. This formula shows that the entanglement of a given state depends on the tensor product decomposition of a Hilbert space. It is shown that the set of separable states is narrower than in the two-qubit case. Moreover, there exist states which are separable with respect to all tensor product decompositions of the Hilbert space.

PACS numbers: 03.67.Mn, 03.65.Ud

Entanglement is the key notion of quantum information theory and plays a significant role in most of its applications. The entanglement of a physical system is always relative to a particular set of experimental capabilities (see, e.g., [1, 2]), which is connected with decompositions of the system into subsystems. From the theoretical point of view this is closely related to possible choices of the tensor product decomposition (TPD) of the Hilbert space of the system. As a consequence, the following question arises: how entangled is a given state with respect to a particular TPD?

In this letter we discuss the problem of the choices of TPD in a system of two fermions, neglecting their spatial degrees of freedom and modifying the tensor product in the rings of operators because of anticommuting canonical variables. We show that TPDs are connected with each other by Bogoliubov transformations of creation and annihilation operators. We also study the behaviour of the entanglement of the system under these transformations. The importance of such an investigation can be illustrated for example by the fact that the Bogoliubov transformations used in the derivation of the Unruh effect also lead to the change of entanglement [3]. A different approach to the entanglement in the system of two identical

0305-4470/05/060079+08$30.00

fermions, based on the asymmetric decomposition of the algebra generated by $a_i, a_i^\dagger$ $(i = 1, 2)$ into a tensor product of two subalgebras was taken up in [4]. Some aspects of the entanglement for a two-fermion system were also discussed in [5].

The theory of entanglement can be seen as the general theory of state transformations that can be performed on multipartite systems, with the restriction that only local operations and classical communications (LOCC) can be implemented [6]. For the same reason, it was expected that additional restrictions should lead to new interesting physical effects and applications. Recently, it has been shown that such a restriction can be given by a superselection rule (SSR) [7, 8].

In this work we restrict the set of physical states of the composite system by the requirement that we prohibit superpositions of fermions and bosons. This leads us to the SSR that is a weaker restriction (i.e., it admits a larger set of states) than the SSR based on the conservation of the number of particles [7]. Moreover, we find the entanglement of formation taking into account the restriction imposed by our SSR.

Let us consider the Hilbert space $\mathscr{H} \cong \mathbb{C}^4$ with an orthonormal basis $\{|m, n\rangle\}_{m,n=0,1}$. With this basis we associate the following two operators:

$$a_1 = |0, 0\rangle\langle 1, 0| - |0, 1\rangle\langle 1, 1|, \tag{1a}$$

$$a_2 = |0, 0\rangle\langle 0, 1| + |1, 0\rangle\langle 1, 1|. \tag{1b}$$

One can easily check that these operators and their Hermitian conjugations fulfil the following relations:

$$\{a_i, a_j\} = 0, \qquad \left\{a_i, a_j^\dagger\right\} = \delta_{ij}, \qquad i, j = 1, 2, \tag{2}$$

where $\{.\,,.\}$ stands for anticommutator. Operators $a_i^\dagger$ generate all the basis vectors from the 'vacuum state' $|0, 0\rangle$ via the relations

$$|1, 0\rangle = a_1^\dagger|0, 0\rangle, \tag{3a}$$

$$|0, 1\rangle = a_2^\dagger|0, 0\rangle, \tag{3b}$$

$$|1, 1\rangle = a_2^\dagger a_1^\dagger|0, 0\rangle, \tag{3c}$$

while the vacuum is annihilated by a_i, i.e. $a_i|0, 0\rangle = 0, i = 1, 2$. We use the occupation number basis, i.e., the Fock basis, not the so-called 'computational basis'. Thus with every orthonormal basis we can associate some representation of the algebra (2). On the other hand it is clear that equations (2) can be interpreted as canonical anticommutation relations for a two-fermion system.

Every two orthonormal bases in $\mathscr{H}$ are connected by some unitary transformation belonging to the group $U(4)$. In the ring of operators these changes of bases are related to Bogoliubov transformations of creation and annihilation operators which will be discussed later on.

One can naively expect that, as in the bosonic case, the operators a_1 and a_2 should have the form $a \otimes \mathrm{id}$ and $\mathrm{id} \otimes a$, respectively, where a is an annihilation operator for a single fermion acting in $\mathbb{C}^2$, that is

$$a\,|0\rangle = 0, \qquad a\,|1\rangle = |0\rangle, \tag{4a}$$

$$a^\dagger|0\rangle = |1\rangle, \qquad a^\dagger\,|1\rangle = 0, \tag{4b}$$

$$\{a, a^\dagger\} = \mathrm{id}, \tag{4c}$$

and id denotes the identity operator. However this is not the case because $a \otimes \mathrm{id}$ and $\mathrm{id} \otimes a$ necessarily commute so they cannot fulfil the canonical anticommutation relations (2). In order to construct a_1, a_2 out of the single annihilation operator a and to provide a natural tensor product interpretation of basis vectors as $|m, n\rangle = |m\rangle \otimes |n\rangle$ we have to modify only the tensor product of the operators acting in $\mathbb{C}^2$. Hereafter we will denote the new tensor multiplication by the usual symbol $\otimes$. Such a modified tensor product is defined by the graded (supersymmetric) multiplication rule

$$(A \otimes \mathfrak{b})(\mathfrak{a} \otimes B) = (-1)^{F(\mathfrak{a})F(\mathfrak{b})} A\mathfrak{a} \otimes \mathfrak{b}B, \tag{5}$$

where $\mathfrak{a}$, $\mathfrak{b}$ are monomials in $a, a^\dagger$, i.e. $\mathfrak{a}, \mathfrak{b} \in \{\mathrm{id}, a, a^\dagger, aa^\dagger, a^\dagger a\}$, A, B are arbitrary operators acting in $\mathbb{C}^2$ and the 'fermion number' $F(\mathfrak{a})$ is equal to the number of creation operators minus the number of annihilation operators building the monomial $\mathfrak{a}$, i.e. $F(\mathrm{id}) = 0$, $F(a^\dagger) = -F(a) = 1$, $F(aa^\dagger) = F(a^\dagger a) = 0$. Consequently the Hermitian conjugation in this tensor product is of the form

$$(\mathfrak{a} \otimes \mathfrak{b})^\dagger = (-1)^{F(\mathfrak{a})F(\mathfrak{b})}(\mathfrak{a}^\dagger \otimes \mathfrak{b}^\dagger). \tag{6}$$

The tensor multiplication introduced above is a special case of a more general structure known in mathematical physics as the braided tensor product [9]. As we can see from relation (5), the new braided tensor product for monomials even in $a, a^\dagger$ behaves like the standard tensor product.

Finally, the relationship between the tensor product of operators and the tensor product of vectors is given by

$$(\mathrm{id} \otimes \mathrm{id})(|m\rangle \otimes |n\rangle) = |m\rangle \otimes |n\rangle, \tag{7a}$$

$$(a \otimes \mathrm{id})(|m\rangle \otimes |n\rangle) = (-1)^n a\,|m\rangle \otimes |n\rangle, \tag{7b}$$

$$(a^\dagger \otimes \mathrm{id})(|m\rangle \otimes |n\rangle) = (-1)^n a^\dagger\,|m\rangle \otimes |n\rangle, \tag{7c}$$

$$(\mathrm{id} \otimes a)(|m\rangle \otimes |n\rangle) = |m\rangle \otimes a|n\rangle, \tag{7d}$$

$$(\mathrm{id} \otimes a^\dagger)(|m\rangle \otimes |n\rangle) = |m\rangle \otimes a^\dagger|n\rangle. \tag{7e}$$

Now the annihilation and creation operators acting in the space $\mathbb{C}^2 \otimes \mathbb{C}^2$ and satisfying (2) take the desired form

$$a_1 = a \otimes \mathrm{id}, \qquad a_2 = \mathrm{id} \otimes a, \tag{8a}$$

$$a_1^\dagger = a^\dagger \otimes \mathrm{id}, \qquad a_2^\dagger = \mathrm{id} \otimes a^\dagger. \tag{8b}$$

Note that in the above equations $\otimes$ denotes the new tensor multiplication thus equations (4*a*)–(7*e*) imply that operators (8*a*)–(8*b*) fulfil the canonical anticommutation relations (2). In particular, the matrix elements of operators (8*a*)–(8*b*) and (1*a*)–(1*b*) are identical in the basis $\{|m, n\rangle\}_{m,n=0,1}$.

Similarly, as in the case of quantum theory of fermionic fields in the system under consideration, observables are restricted to combinations of even products of creation and annihilation operators. In particular the local observables are combinations of $\mathrm{id} \otimes \mathrm{id}$ and $N_1 = a^\dagger a \otimes \mathrm{id}$ or $\mathrm{id} \otimes \mathrm{id}$ and $N_2 = \mathrm{id} \otimes a^\dagger a$. It is implied by the SSR related to the requirement that the operator $(-1)^{\hat{F}}$, where $\hat{F}$ is the fermion number operator, should commute with all observables [10]. It means that superpositions of bosons and fermions are forbidden. In the basis (3*a*)–(3*c*) $(-1)^{\hat{F}} = \mathrm{diag}\{1, -1, -1, 1\}$. Alternatively, this SSR is a consequence of the requirement that the squared time reflection operator must commute with all observables (see, e.g., [11]). Indeed, the antiunitary time inversion operator is defined here as follows

$$\mathsf{T}a_1\mathsf{T}^{-1} = a_2, \qquad \mathsf{T}a_2\mathsf{T}^{-1} = -a_1, \tag{9}$$

$$\mathsf{T}\,|0,0\rangle = |0,0\rangle. \tag{10}$$

Thus $\mathsf{T}^2 = (-1)^{\hat{F}}$. Due to the SSR the density matrix has to commute with $(-1)^{\hat{F}}$, so the general state of this system is represented by the following density matrix:

$$\rho = \begin{pmatrix} w_1 & 0 & 0 & b_1 \\ 0 & w_2 & b_2 & 0 \\ 0 & b_2^* & v_2 & 0 \\ b_1^* & 0 & 0 & v_1 \end{pmatrix}, \tag{11}$$

where $w_i, v_i \geqslant 0$, $\sum_{i=1}^{2}(w_i + v_i) = 1$ and $|b_i|^2 \leqslant w_i v_i$, $i = 1, 2$. Consequently, possible states of subsystems obtained from (11) by partial traces are

$$\rho_1 = \begin{pmatrix} w_1 + v_2 & 0 \\ 0 & w_2 + v_1 \end{pmatrix}, \tag{12a}$$

$$\rho_2 = \begin{pmatrix} w_1 + w_2 & 0 \\ 0 & v_1 + v_2 \end{pmatrix}. \tag{12b}$$

Note that the diagonal form of (12*a*)–(12*b*) conforms with the SSR in spaces of subsystems. Moreover, the states (12*a*)–(12*b*) exhaust all possible states of the subsystems. Therefore our subsystems are independent in the sense of the definition of the algebraic independence of subsystems [4, 12]. This independence is due to the SSR (compare [4] where it was shown that in general algebras of observables of two identical fermions are nonindependent). The natural question arises: what is the form of the separable states for this system? According to Werner's definition [13] the state is separable if it can be written in the form $\rho = \sum_i p_i \rho_1^i \otimes \rho_2^i$, where ρ_1^i and ρ_2^i are admissible states of subsystems and $\sum_i p_i = 1$, $p_i \geqslant 0$. Therefore, taking into account that ρ_1^i and ρ_2^i are of the form (12*a*)–(12*b*), the separable states have the surprisingly simple diagonal form

$$\rho_{\text{sep}} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}, \tag{13}$$

with $\sum_i \lambda_i = 1$, $\lambda_i \geqslant 0$. Consequently, nondiagonal density matrices are nonseparable. Thus in this case the standard method of calculating entanglement measures should be taken with care. Indeed, as an example let us consider the Werner state [13, 14]

$$\rho_W = \begin{pmatrix} \frac{1+\gamma}{4} & 0 & 0 & \frac{\gamma}{2} \\ 0 & \frac{1-\gamma}{4} & 0 & 0 \\ 0 & 0 & \frac{1-\gamma}{4} & 0 \\ \frac{\gamma}{2} & 0 & 0 & \frac{1+\gamma}{4} \end{pmatrix}, \qquad \gamma \in [-1/3, 1], \tag{14}$$

which belongs to the admissible states (11). The Wootters concurrence [15] of this state is equal to zero for $\gamma \in [-1/3, 1/3]$, therefore for two qubits the Werner state is separable for such values of γ. On the other hand, in our case this state is separable only when $\gamma = 0$. Thus the Wootters concurrence does not define entanglement measure in our case.

Instead, let us calculate directly the entanglement of formation [6], i.e.

$$E(\rho) = \min \sum_i p_i S\big(\rho_A^i\big), \tag{15}$$

where $S(\rho_A) = -\operatorname{Tr} \rho_A \log_2 \rho_A$ is the von Neumann entropy and the minimum is taken over all the possible realizations of the state $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ with $\rho_A^i = \operatorname{Tr}_B(|\psi_i\rangle\langle\psi_i|)$. Taking into account the special form of the density matrix (11) we can find the explicit formula for the entanglement of formation

$$E(\rho) = \sum_{i=1}^{2} (w_i + v_i) S_i \tag{16}$$

where

$$S_i = \begin{cases} 0 & \text{if } w_i = v_i \quad \text{and} \quad b_i = 0 \\ -\dfrac{1}{2}\left[(1-\xi_i)\log_2 \dfrac{1-\xi_i}{2} + (1+\xi_i)\log_2 \dfrac{1+\xi_i}{2}\right] & \text{otherwise} \end{cases} \tag{17}$$

and

$$\xi_i = \frac{w_i - v_i}{\sqrt{(w_i - v_i)^2 + 4|b_i|^2}}. \tag{18}$$

It is interesting that a formula similar to (17) was obtained in [16] for the so-called correlational entropy of the two-level system. Note that the maximal value of $E(\rho)$ is equal to 1. In the case of the Werner state (14) the entanglement of formation (16) is

$$E(\rho_W) = \begin{cases} \dfrac{1+\gamma}{2} & \text{if } \gamma \neq 0, \\ 0 & \text{if } \gamma = 0. \end{cases} \tag{19}$$

Thus, as expected, $E(\rho_W) \neq 0$ for entangled (nondiagonal) states and $E(\rho_W) = 0$ for a separable (diagonal) state. For $\gamma = 1$ we have the maximally entangled Werner state. Note that the restriction of admissible states by the SSR implies that in our case we have no asymmetry in the definition of the entanglement of formation, in contrast to observations of [4].

Let us consider the problem of decomposition of our system into two subsystems. Such a decomposition corresponds to the different choices of canonical variables $a_i, a_i^\dagger$. This is extremely important because each choice of $a_i, a_i^\dagger$ defines in the Hilbert space $\mathscr{H}$ the corresponding tensor product structure (5)–(7*e*) such that the creation and annihilation operators take the form analogous to (8*a*)–(8*b*). Each TPD defines a set of local observables of the form $A \otimes \mathrm{id}$ and $\mathrm{id} \otimes B$ (cf the discussion after (8*a*)–(8*b*)). Moreover, the notion of a local observer is determined by his experimental access to local observables (see, e.g., [1]).

Different choices of canonical variables $a_i, a_i^\dagger$ are connected by transformations which preserve the canonical anticommutation relations (2) (Bogoliubov transformations[1]). Therefore Bogoliubov transformations give us all possible decompositions of the two-fermion system into two subsystems (two fermions). Such decompositions of the system correspond to the tensor product decompositions of the space $\mathscr{H} \cong \mathbb{C}^2 \otimes \mathbb{C}^2$, appropriate to the definition of the subsystems. In the case under consideration the problem of finding all possible TPDs consistent with our SSR is equivalent to determining all possible Bogoliubov transformations commuting with the superselection operator $\mathsf{T}^2 = (-1)^{\hat{F}}$.

[1] By Bogoliubov transformations we mean here all transformations of creation and annihilation operators (linear as well as nonlinear) which do not change the canonical anticommutation relations.

Let us note first that operators $a_i, a_i^\dagger$ in every orthonormal basis can be represented in the form (3a)–(3c) and vice versa such operators define an orthonormal basis via (3a)–(3c). Thus different choices of these operators are connected with different choices of orthonormal bases in the Hilbert space. Therefore

$$a_i' = U a_i U^\dagger, \tag{20}$$

where U is an unitary matrix. As we have mentioned above, the consistency with the SSR means that U commutes with $\mathsf{T}^2 = (-1)^{\hat{F}} = \mathrm{diag}\{1, -1, -1, 1\}$. So U can be represented as the following product of unitary matrices:

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha^* & -\beta & 0 \\ 0 & \beta^* & \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta & 0 & 0 & -\omega^* \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega & 0 & 0 & \zeta^* \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-\mathrm{i}\chi} \end{pmatrix}, \tag{21}$$

with $|\alpha|^2 + |\beta|^2 = 1$ and $|\zeta|^2 + |\omega|^2 = 1$, where we took into account the fact that equation (20) determines U up to an overall phase. Thus all Bogoliubov transformations admissible by the SSR form the group $SU(2) \otimes U(2)$. Applying these transformations to the explicit matrix form of $a_i, a_i^\dagger$ calculated from (1a)–(1b), one can show that in the ring of creation and annihilation operators the transformations (20) are realized as

- $SU(2)$ transformations which do not mix creation and annihilation operators

$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \tag{22}$$

- $SU(2)$ transformations which mix creation and annihilation operators

$$\begin{pmatrix} a_1' \\ a_2'^\dagger \end{pmatrix} = \begin{pmatrix} \zeta & \omega \\ -\omega^* & \zeta^* \end{pmatrix} \begin{pmatrix} a_1 \\ a_2^\dagger \end{pmatrix}, \tag{23}$$

- nonlinear one-parameter transformations

$$a_1' = a_1\, \mathrm{e}^{\mathrm{i}\chi N_2} = a_1[1 + (\mathrm{e}^{\mathrm{i}\chi} - 1)N_2], \tag{24a}$$

$$a_2' = a_2\, \mathrm{e}^{\mathrm{i}\chi N_1} = a_2[1 + (\mathrm{e}^{\mathrm{i}\chi} - 1)N_1]. \tag{24b}$$

Note that (24a)–(24b) for $\chi = \pi$ are the so-called Klein–Wigner transformations (cf [4]). The Bogoliubov transformations which lead to physically distinguishable TPDs should change the local observables $N_i = a_i^\dagger a_i$. Therefore such transformations have the form (22) with both $\alpha \neq 0$, $\beta \neq 0$ and/or (23) with both $\zeta \neq 0$, $\omega \neq 0$.

Now, the natural question arises: what does the same state look like for local observers connected with different TPDs? The answer is quite obvious: if their TPDs are connected by Bogoliubov transformations then density matrices representing the state are connected by similarity transformations, i.e. $\rho' = U\rho U^\dagger$. However, in general, such transformations change the entanglement measure $E(\rho)$, i.e. entanglement depends on the choice of TPD (and hence the local observers). In particular, for any state, there exists a pair of observers for whom this state is separable, since the density matrix (11) can always be diagonalized by means of the transformations (21). We point out that there exists a class of *superseparable* states $\rho_{\mathrm{ss}} = \frac{1}{2}\mathrm{diag}\{s, 1-s, 1-s, s\}$, $s \in [0, 1]$, which are separable for every observer. Note also that in the case of two qubits only one superseparable state exists, namely the maximally mixed state $\rho_0 = \frac{1}{4}I$.

Now we show that it is possible to construct dynamics consistent with our SSR. For such a dynamics admissible TPDs are related to symmetries of the Hamiltonian. An example of that dynamics is the Thirring model [17] in $(1+0)$-dimensional space–time describing a fermionic quantum mechanical system. The corresponding Lagrangian is of the form

$$L = \sum_{i=1}^{2} \left(i\psi_i^\dagger \partial_t \psi_i - m\psi_i^\dagger \psi_i \right) - \lambda \left(\sum_{i=1}^{2} \psi_i^\dagger \psi_i \right)^2 . \tag{25}$$

The solutions of the equations of motion derived from the Lagrangian (25) are

$$\psi_i(t) = a_i\, \mathrm{e}^{-\mathrm{i}t(m+\lambda+2\lambda N_j)}, \qquad i \neq j, \tag{26}$$

where the time-independent operators a_i and $a_i^\dagger$ satisfying (2) can be represented in the form (8*a*)–(8*b*). The Hamiltonian of this system is

$$H = (m+\lambda)(N_1 + N_2) + 2\lambda N_1 N_2 \tag{27}$$

and describes two fermionic oscillators with the quartic interaction term. Note that $\mathsf{T}^2 = (-1)^{\hat{F}}$ commutes with H, thus Thirring model dynamics undergoes our SSR.

In the special case of $\lambda = -\frac{1}{2}m$ all the Bogoliubov transformations (22)–(24*b*) form the symmetry group of H, i.e. $H(a, a^\dagger) = H(a', a'^\dagger)$. Thus, this symmetry group gives us a freedom with a choice of a concrete decomposition of the system into two subsystems. Consequently the related TPDs are connected by the Bogoliubov transformations (22)–(24*b*).

In conclusion, we have investigated the dependence of entanglement for a two-fermion system on tensor product decompositions in the presence of the superselection rule. We have shown that the Wootters concurrence is not a proper entanglement measure in this case. The crucial point in finding an explicit form of entanglement of formation for such a system was determining the states of subsystems, admissible by the superselection rule. We would like to stress that these states *are not* qubit states. It is interesting that the set of separable states is narrower than in the two-qubit case, namely it consists of only the states represented by diagonal density matrices. Moreover, we found the class of superseparable states, i.e. the states which are separable with respect to all tensor product decompositions of the Hilbert space.

Acknowledgments

We would like to thank P Horodecki and T Brzeziński for helpful discussions. This paper has been partially supported by the Polish Ministry of Scientific Research and Information Technology under the grant no PBZ-MIN-008/P03/2003.

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